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Suzuki Groups and 2-(v, k, 1) Designs

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If one lets $\mathcal{D}$ be a 2-$(v, k, 1)$ design with $G \leq \mathrm{Aut}(\mathcal{D})$ block-primitive, then G does not have a Suzuki group as its socle.

1. INTRODUCTION

A 2-$(v, k, 1)$ design $\mathcal{D} = (\Omega, \mathcal{B})$ is a system consisting of a finite set Ω of v points and a collection $\mathcal{B}$ of k-subsets of Ω, called blocks, such that any 2-subset of Ω is contained in precisely one block. We assume throughout that $2 < k < v$.

Let $G \leq \mathrm{Aut}(\mathcal{D})$ be a group of automorphisms of a 2-$(v, k, 1)$ design $\mathcal{D}$. The group G is said to be *block-transitive* (*block-primitive*, respectively) on $\mathcal{D}$ if G is transitive (primitive, respectively) on $\mathcal{B}$. Group G is said to be *point-transitive* (*point-primitive*, respectively) on $\mathcal{D}$ if G is transitive (primitive, respectively) on Ω. A *flag* of $\mathcal{D}$ is a pair consisting of a point and a block containing this point. G is *flag-transitive* on $\mathcal{D}$ if G is transitive on the set of flags of $\mathcal{D}$. The following results are well known: if G is block transitive, then G is also point-transitive (see [1]); if G is flag-transitive, then G is point-primitive (see [8]); if G acts block primitively on a finite projective plane, then G is point-primitive (see [11]). Doyen and Delandtsheer [5] conjecture that if G is block-primitive, then G is also point-primitive.

It is known that the conjecture is true under any of the following hypotheses:

(1) $\mathcal{D}$ is a finite projective plane;
(2) $k/(k, v) \leq 4$;
(3) $v > \frac{[k(k-1)/2-1]^2}{2}$;
(4) $k \leq 40$;
(5) G has a subgroup acting regularly on Ω;
(6) the rank of G acting on $\mathcal{B}$ does not exceed 7.

Results (1), (3), (5) and (6) can be found in [5]; (2) and (4) in [5, 7, 15]. Delandtsheer [6] proved the following result: If one lets $\mathcal{D}$ be a 2-$(v, k, 1)$ design other than a projective plane and $G \leq \mathrm{Aut}(\mathcal{D})$ be block-primitive, then G is almost simple, i.e., there is a non-abelian simple T such that $T \trianglelefteq G \leq \mathrm{Aut}(T)$. Li and Liu proved the conjecture holds if $T = \mathrm{soc}(G)$ is an alternating group A_n or a sporadic simple group, respectively. In [17], Zhou *et al*. proved the above conjecture is true if $T \cong {}^2G_2(q)$.

In this paper, we prove the following theorem.

THEOREM 1.1. *If one lets $\mathcal{D}$ be a* 2-$(v, k, 1)$ *design and* $G \leq \mathrm{Aut}(\mathcal{D})$ *be block-primitive, then the socle of G is not isomorphic to $Sz(q)$.*

Thus this supports Doyen–Delandtsheer's conjecture.

The second section describes the notation and contains a lot of preliminary results regarding the simple Suzuki groups $Sz(q)$ and 2-$(v, k, 1)$ designs.

In the third section, we will give the proof of Theorem 1.1.

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2. Preliminary Results

In this section, we begin by restating some fundamental properties of the simple Suzuki groups $Sz(q)$.

We put $q = 2^{2m+1}$, $t = 2^m$ and $G = Sz(q)$. By Theorem 3.10 of Chapter XI of [9], G possesses cyclic subgroups U_1 and U_2 of orders $q + 2t + 1$ and $q - 2t + 1$, respectively. U_1 and U_2 are Hall subgroups of G. By Lemma 3.1 of Chapter XI of [9], G possesses subgroups F and H, where F is a 2-group of exponent 4, class 2 and order q^2. H is isomorpic to $GF(q)^\times$, a multiplicative group of $GF(q)$.

Lemma 2.1 ([9, Chapter XI, Lemma 3.12]). *Every maximal subgroup of G is conjugate to one of the following:*

(1) $Sz(a)$, $a^i = 2^{2m+1}$, *i a prime;*
(2) FH;
(3) $N_G(H)$;
(4) $N_G(U_i)$, $i = 1, 2$.

Conversely, there is exactly one class of maximal subgroups of G for each case in Lemma 2.1. Since $|Sz(q)| = q^2(q^2 + 1)(q - 1)$ and $q^2 + 1 = (q + 2t + 1)(q - 2t + 1)$, and q^2, $q+2t+1$, $q-2t+1$, $q-1$ are mutually prime, the Sylow p-subgroups of $Sz(q)$ are conjugate to the subgroups of F, H, U_1 or U_2.

Lemma 2.2 ([9, Chapter XI]). *For any* $g \in G$, *we have:*

(1) *if* $F \neq F^g$, *then* $F \cap F^g = \{1\}$;
(2) *if* $H \neq H^g$, *then* $H \cap H^g = \{1\}$, *and for all* $1 \neq h \in H$, $C_G(h) = H$;
(3) *if* $U_i \neq U_i^g$, *then* $U_i \cap U_i^g = \{1\}$, *and for all* $1 \neq u \in U_i$, $C_G(u) = U_i$, *where* $i = 1$ *and* 2.

Lemma 2.3. *Let P be a Sylow p-subgroup of G:*

(1) *if* $P = F$, *then* $N_G(P) = FH$;
(2) *if* $P \leq H$, *then* $N_G(P) = N_G(H)$;
(3) *if* $P \leq U_i$, *then* $N_G(P) = N_G(U_i)$, *where* $i = 1$ *and* 2.

Proof. (1) By Chapter XI, Lemma 3.1 of [9], $H \leq N_G(F)$, thus $FH \leq N_G(F)$. Since FH is a maximal subgroup of G, it follows that $N_G(F) = FH$.

(2) For any $g \in N_G(P)$, $P \leq H \cap H^g$. By Lemma 2.2, $H = H^g$, this means $g \in N_G(H)$. Thus $N_G(P) \leq N_G(H)$. Since the group H is cyclic, any element of G which normalizes H will normalize any Sylow p-subgroup of H. Thus $N_G(P) = N_G(H)$.

(3) The proof is similar to that of (2).

Let $\mathcal{D}$ be a 2-$(v, k, 1)$ design. Ω is the set of points of $\mathcal{D}$, $\mathcal{B}$ is the set of blocks of $\mathcal{D}$, b is the number of blocks of $\mathcal{D}$ and r is the number of blocks of $\mathcal{D}$ through a given point of Ω. Then

$$b = \frac{v(v-1)}{k(k-1)}, \qquad r = \frac{(v-1)}{(k-1)}.$$

Let

$$b_1 = (b, v), \qquad b_2 = (b, v-1), \qquad k_1 = (k, v) \qquad \text{and} \qquad k_2 = (k, v-1).$$

Obviously,

$$k = k_1k_2, \qquad b = b_1b_2, \qquad r = b_2k_2 \qquad \text{and} \qquad v = b_1k_1.$$

Let B be a block of $\mathcal{D}$. Then, G_B will be the block stabilizer and $G_{(B)}$ the pointwise stabilizer of the block.

If G is a block-transitive automorphism group of $\mathcal{D}$, and not primitive on Ω. We define $\mathcal{C}$ as follow: $\mathcal{C}$ is a set of some non-trivial partition of Ω which is preserved by G and on which G acts primitively. Let c be the cardinality of $\mathcal{C}$ and d the common size of the classes in $\mathcal{C}$. Therefore, $v = cd$. □

LEMMA 2.4. *(1)* (*Lemma* 2 *of* [7]). *There exist positive integers x and y, such that $c = xb_2 + 1$ and $d = yb_2 + 1$.*

(2) (*Lemma* 4 *of* [6]). *There exists a prime number p dividing b but not v, and $b > v$.*

The following Lemmas are very useful for our proof of Theorem 1.1.

LEMMA 2.5 (LEMMA 2 OF [4]). *Let G act as a block-transitive automorphism group of a* 2-$(v, k, 1,)$ *design. Let B be a block and H a subgroup of G_B. Assume that H satisfies the two following conditions:*

(i) $|\mathrm{Fix}(H) \cap B| \geq 2$ *and;*
(ii) if $K \leq G_B$ and $|\mathrm{Fix}(K) \cap B| \geq 2$ and K is conjugate to H in G then H is conjugate to K in G_B.

Then either (a) $\mathrm{Fix}(H) \subseteq B$ *or (b) the induced structure on* $\mathrm{Fix}(H)$ *is a* 2-$(v_0, k_0, 1)$ *design where $v_0 = |\mathrm{Fix}(H)|$, $k_0 = |\mathrm{Fix}(H) \cap B|$. Furthermore, $N_G(H)$ acts as a block-transitive group on this design.*

LEMMA 2.6. *Let G act as a block-transitive automorphism group of a* 2-$(v, k, 1)$ *design. Let B be a block and v even. Assume that there exists a* 2*-subgroup P of G_B such that* $\mathrm{Fix}(P) \subseteq B$. *Then $k \mid v$ and G is flag-transitive.*

PROOF. Let i be an involution in P. As v is even, then k is even. For any block $B' \neq B$, if $B'^i = B'$, then B' has no point fixed by i. Suppose B_1 and B_2 are two blocks fixed by i. Then $B_1 \cap B_2 = \emptyset$. In fact, if $\alpha \in B_1 \cap B_2$, then α is fixed by i. Hence i fixes at least one other point of B_1, and this implies that $B = B_1$. Similarly, we may derive $B = B_2$. Consider the cycle decomposition of i acting on Ω, we have $|\mathrm{Fix}_{\mathcal{B}}(\langle i\rangle)| = (v - k)/k + 1 = v/k$, this means $k|v$, where $\mathrm{Fix}_{\mathcal{B}}(\langle i\rangle)$ denotes the set of blocks fixed by i. By [3], G is flag-transitive. □

LEMMA 2.7. *Let G act as a block-transitive automorphism group of a* 2-$(v, k, 1)$ *design. Let B be a block and let i be an involution of G_B. Assume that G_B has a unique conjugate class of involutions. If* $|\mathrm{Fix}(\langle i\rangle) \cap B| \geq 2$ *and v is even, then G is flag-transitive or the induced structure on* $\mathrm{Fix}(\langle i\rangle)$ *is a* 2-$(v_0, k_0, 1)$ *design where $v_0 = |\mathrm{Fix}(\langle i\rangle)|$, $k_0 = |\mathrm{Fix}(\langle i\rangle) \cap B|$. Furthermore, $N_G(\langle i\rangle)$ acts as a block-transitive group on this design.*

PROOF. Set $H = \langle i\rangle$. Clearly H satisfies the hypothesis of Lemma 2.5. If $\mathrm{Fix}(H) \subseteq B$ then G is flag transitive by Lemma 2.6. If $\mathrm{Fix}(H) \not\subseteq B$, then by Lemma 2.5, Lemma 2.7 is true. □

LEMMA 2.8. *Let $G \leq \mathrm{Aut}(\mathcal{D})$, G act block-transitively on $\mathcal{D}$. If there exists a prime number p such that $p|b$ but $p \vee v$, then for some $\alpha \in \Omega$, $N_G(P) \leq G_\alpha$, where P is a Sylow p-subgroup of G.*

PROOF. (i) P does not fix any block of $\mathcal{D}$.

Since $b = |G|/|G_B|$ and $p|b$, it follows that $P \not\leq G_B$ for any $B \in \mathcal{B}$.

(ii) P fixes exactly one point of Ω.

Suppose that P does not fix any point of Ω, then for any point β of Ω, $|\beta^P| = |P : P_\beta| > 1$. Since $v = \sum |\beta^P|$, $p|v$. This conflicts with the hypothesis. Hence P fixes exactly one point by (i).

(iii) $N_G(P) \leq G_\alpha$.

For any $g \in N_G(P)$, $\alpha^{gP} = \alpha^{Pg} = \alpha^g$. So α^g is fixed by P. By (ii), $\alpha = \alpha^g$, thus $N_G(P) \leq G_\alpha$. □

LEMMA 2.9 ([12, 13]). *Let G be a finite group and M a maximal subgroup of G not containing* $\mathrm{Soc}(G)$. *If* $\mathrm{Soc}(G) = Sz(q)$, *where* $q = 2^{2m+1}$ *and* $m > 0$, *then* $M \cap Sz(q)$ *is a maximal subgroup of* $Sz(q)$.

In fact, it is not difficult to prove the above lemma.

3. PROOF OF THEOREM 1.1

First we prove the following proposition.

PROPOSITION 3.1. *Let $\mathcal{D}$ be a* 2-$(v, k, 1)$ *design and* $G \leq \mathrm{Aut}(\mathcal{D})$ *be block-primitive. If* $Sz(q) \trianglelefteq G \leq \mathrm{Aut}(Sz(q))$, *then G is also point-primitive.*

PROOF. Since G is block-primitive, we have that G_B is a maximal subgroup of G for any block B of $\mathcal{D}$. By $Sz(q) \trianglelefteq G$ we get $Sz(q)$ is block-transitive. Thus $G_B \cap Sz(q) \neq Sz(q)$. By Lemma 2.9, we may assume that $G = Sz(q)$.

Suppose that there exist counterexamples $(\mathcal{D}, G)$. Then there is a non-trivial partition $\mathcal{C} = \{C = C_1, C_2, \ldots, C_c\}$, $|C_i| = d, i = 1, 2, \ldots, c$, such that G acts primitively on $\mathcal{C}$, and this action is faithful (see [6]). Thus G_C is a maximal subgroup of G and G_C is one of the groups of Lemma 2.1. By Lemma 2.4, there is a prime p, such that $p|b$ but $p \vee v$. Hence by Lemma 2.8 there exists a Sylow p-subgroup P of G, such that $N_G(P) \leq G_\alpha < G_C$.

(1) If $G_C \cong Sz(a)$, $q = a^i$ and i is a prime, then by Lemmas 2.3 and 2.8, G_C contains H or U_1 or U_2. This is impossible.

(2) If $G_C \cong FH$, then $p = 2$ or p divides $q - 1$. Thus $N_G(P) \cong FH$ or $N_G(H)$ by Lemma 2.3. Since FH and $N_G(H)$ are the maximal subgroups of G, we have $G_\alpha = G_C$, a contradiction.

(3) If $G_C \cong N_G(H)$, then p divides $q - 1$. By Lemma 2.3, we have $N_G(P) = N_G(H)$ and hence $G_\alpha = G_C$. Again we get a contradiction.

(4) If $G_C \cong N_G(U_i)$, where $i = 1, 2$, then p divides $q + (-1)^i 2t + 1$ and $P \leq U_i$. Therefore $N_G(P) = N_G(U_i)$ by Lemma 2.3, which conflicts with $G_\alpha < G_C$.

By (1)–(4) above we conclude that G_α is a maximal subgroup of G. Hence G acts primitively on Ω and so does G, which gives rise to a contradiction. Thus the assertion of the proposition holds. □

Now we start to prove our main theorem.

Suppose $\mathrm{Soc}(G) \cong Sz(q)$, then by Lemma 2.9, we may assume that $G = Sz(q)$. Thus G_α and G_B are all maximal subgroups of G and so they occur in Lemma 2.1. Clearly $\mathcal{D}$ is not a projective plane (see [11]). It follows that $b > v$, i.e., $|G_B| < |G_\alpha|$. By [10], $G_\alpha \ncong FH$. If $G_\alpha \cong Sz(a)$, where a satisfies $q = a^m$, m a prime, then by Lemma 2.8, there always exists a

Sylow p-subgroup P of G, where p is odd, such that $N_G(P) \leq G_\alpha$. According to Lemma 2.3, G_α contains subgroups H or U_1 or U_2 of G. This is impossible. Thus $G_\alpha \ncong Sz(a)$.

First we eliminate the case $G_B \cong Sz(a)$.

In this case, $G_\alpha \cong N_G(U_1)$, $N_G(U_2)$ or $N_G(H)$. In order to calculate $|\mathrm{Fix}(\langle i\rangle)|$, we consider the pair (i, G_α), where i is an involution of G such that i is contained in G_α. Since G has a unique conjugacy class of involutions (see [16]) by counting pairs on two ways we see that

$$|G : G_\alpha| \cdot e(G_\alpha) = e(G) \cdot N,$$

where $e(G)$ denotes the number of involutions of G, and N denotes the number of conjugates of G_α which contains i, it is equal to the number of points fixed by i, i.e., $N = |\mathrm{Fix}(\langle i\rangle)|$. Hence

$$|\mathrm{Fix}(\langle i\rangle)| = \frac{|G : G_\alpha| \cdot e(G_\alpha)}{e(G)}.$$

By [16],

$$e(G) = |G : C_G(i)| = (q^2 + 1) \cdot (q - 1).$$

Thus

$$|\mathrm{Fix}(\langle i\rangle)| = \frac{q^2 \cdot e(G_\alpha)}{|G_\alpha|}. \tag{$*$}$$

By [16],

$$N_G(U_1) = \langle u, s \mid o(u) = q + 2t + 1, \qquad o(s) = 4,\ u^s = u^q\rangle.$$

s^2u^j are involutions of $N_G(U_1)$, where $j = 1, 2, \ldots, q+2t+1$. Thus $e(N_G(U_1)) = q+2t+1$. Similarly, $e(N_G(U_2)) = q - 2t + 1$ and $e(N_G(H)) = q - 1$.

If $G_\alpha \cong N_G(U_1)$, by $(*)$, $|\mathrm{Fix}(\langle i\rangle)| = q^2/4$. Set $Q = \langle i\rangle$. By the block transitivity of G, we may assume that $Q \leq G_B$. Clearly G_B and $Q = \langle i\rangle$ satisfy the conditions of Lemma 2.7. Then G is flag transitive or the induced structure on $\mathrm{Fix}(Q)$ is a 2-$(q^2/4, k_0, 1)$ design, where $k_0 = |\mathrm{Fix}(Q) \cap B|$. Furthermore, $N_G(Q)$ acts as a block-transitive on this design. By [2] and [14], G is not flag-transitive. Thus the latter holds. Since $N_G(Q) \cong F$, $|N_G(Q)| = q^2$ (see [16]). It follows that $b_0 \mid q^2$, where b_0 is the number of blocks of the above-induced structure. Since $b_0 \geq v_0 = q^2/4$, $b_0 = q^2, q^2/2$ or $q^2/4$. Thus we have the equality

$$4k_0^2 - 4k_0 = \frac{q^2}{4} - 1$$

or

$$2k_0^2 - 2k_0 = \frac{q^2}{4} - 1$$

or

$$k_0^2 - k_0 = q^2/4 - 1.$$

When $m > 1$, these equalities are not satisfied.

Therefore, when $G_B \cong Sz(a)$, then $G_\alpha \ncong N_G(U_1)$.

Similarly, we may prove that if $G_B \cong Sz(a)$, then $G_\alpha \ncong N_G(U_2)$ and $G_\alpha \ncong N_G(H)$. Thus $G_B \ncong Sz(a)$.

Next we discuss the remaining cases of G_α and G_B.

(i) $G_\alpha \cong N_G(U_1)$

We have

$$v = \frac{q^2(q^2 + 1)(q - 1)}{4(q + 2t + 1)} = \frac{q^2(q - 2t + 1)(q - 1)}{4}.$$

By $|G_B| < |G_\alpha|$, $G_B \cong N_G(U_2)$ or $N_G(H)$.

If $G_B \cong N_G(U_2)$, then $b_2 = q + 2t + 1$. Since $b_2 k_2(k-1) = v - 1$, we have

$$(q + 2t + 1)k(k-1) = (q - 2t + 1) \cdot (q^2(q - 2t + 1)(q-1)/4 - 1).$$

When $m > 1$, the left-hand side of this equality is even, but the right-hand side is odd, a contradiction.

If $G_B \cong N_G(H)$, then

$$b_2 = \frac{b}{b_1} = \frac{b}{(b, v)} = 2(q + 2t + 1).$$

But then $(b_2, v) \neq 1$. This is a contradiction.

(ii) $G_\alpha \cong N_G(U_2)$

Clearly,

$$v = \frac{q^2(q + 2t + 1)(q-1)}{4},$$

and $G_B \cong N_G(H)$. Thus $b_2 = 2(q - 2t + 1)$, conflicting with $(b_2, v) = 1$.

(iii) $G_\alpha \cong N_G(H)$

By $|G_B| < |G_\alpha|$, then $G_B \cong Sz(a)$. This conflicts with the statement as above.

To sum up, we have derived a contradiction. Thus G is not isomorphic to $Sz(q)$. This completes the proof of Theorem 1.1.

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